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U. S. ARMY

TRANSPORTATION RESEARCH COMMAND
FORT EUSTIS, VIRGINIA

TRECOM TECHNICAL REPORT 63-56

SOME DYNAMIC ASPECTS OF STABILITY IN LOW-SPEED FLYING MACHINES

Task 1D121401A14203
(Formerly Task 9R38-11-009-03)
Contract DA 44-177-TC-835

November 1963

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US ARMY MOBILITY COMMAND

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In this report, Princeton University investigated the dynamic aspects of stability in low-speed flying machines from the standpoint of time-varying systems.

The results are used in a discussion of direct feedback adjustments, adaptive feedback and programmed feedback adjustments.

This command considers the conclusions made by the contractor to be valid.

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Task 1D121-401-A142-03
(Formerly Task 9R38-11-009-03)
Contract DA 44-177-TC-835
TRECOM Technical Report 63-56
November 1963

SOME DYNAMIC ASPECTS OF STABILITY IN LOW-SPEED FLYING MACHINES

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FOREWORD

The research in this report was conducted in the Instrumentation and Control Laboratory of the Department of Aeronautical Engineering at Princeton University.

The work was done under the sponsorship of the U.S. Army Transportation Research Command under Contract DA 44-177-TC-835. This support is gratefully acknowledged.

CONTENTS

ILLUSTRATIONS

LIST OF SYMBOLS

B	rate of damping variation
D	rate of spring constant variation
K_δ	damping coefficient of the approximating differential equation
K_ω	spring constant of the approximating differential equation
$M_{(\ )} = \frac{1}{I_y} \frac{\partial M_y}{\partial(\ )}$	normalized pitching moment derivative (I_y = moment of inertia)
P	rate of damping variation in the normalized equation
Q	rate of spring constant variation in the normalized equation
Q_1 , Q_2	coefficients in the Taylor series of the normalized spring constant variation
S	Laplace transform variable
t	real time
$t' = \omega_F t$	nondimensional time
u	horizontal velocity (small perturbation variable)
V	horizontal velocity
$V_{F.F.}$	horizontal velocity after completion of the transition ($V_{F.F.} = V_{forward\ flight}$)
w	vertical velocity (small perturbation variable)
$X_{(\ )} = \frac{1}{m} \frac{\partial F_x}{\partial(\ )}$	normalized horizontal force derivative (m = mass of aircraft)

$x(t)$	variable (time varying system)
$y(t')$	variable (normalized time varying system)
$y_a(t')$	variable (normalized approximating constant coefficient system)
$Z_{(\ )} = \frac{1}{m}\frac{\partial F_Z}{\partial(\ )}$	normalized vertical force derivative (m = mass of aircraft)
α	angle of attack (small perturbation variable)
$\beta = B/\omega_F^2$	normalized rate of damping variation
$\mu = D/\omega_F^3$	normalized rate of spring constant variation
δ	elevator angle; unit pulse
$\epsilon(t') = y(t') - y_a(t')$	error of approximation
ζ_F	"frozen" damping ratio
θ	attitude angle (small perturbation variable)
τ	"frozen" time
ω_{Fo}	"frozen" natural frequency
$\omega_F = \omega_{Fo}\sqrt{1 - \zeta_F^2}$	"frozen" damped frequency

SUMMARY

This report is concerned with a linear time varying approximation to the dynamics of low-speed flying machines. Simplifications and approximations are applied widely in order to emphasize essential aspects.

The range of time variation is described in terms of frozen system loci of the roots corresponding to the predominant mode of a system. The rate of the time variation is described in terms of the deviation from the frozen system approximation. An analog computer study was made to specify quantitatively those rates of time variation which cannot be considered as slow.

The longitudinal dynamics of VTOL aircraft is studied as an example in rather general terms. Approximations and the application of root locus methods in terms of the most significant stability derivatives lead to a construction describing the behavior of the oscillatory roots during transition.

The results are used in a discussion of the following variable feedback configurations: direct feedback adjustments, adaptive feedback, and programmed feedback adjustments.

CONCLUSIONS

Two distinct aspects of time varying systems are the range and the rate of time variation. The question of when the time variation should be considered "fast" is answered on the basis of practical considerations by means of a computer study. The rate of time variation should be considered fast if, during the time of one period of the predominant mode, the spring constant variation is equal to (or more than) 50% and/or the damping coefficient variation is equal to (or more than) .5. In these cases the system response is markedly different from that obtained by the frozen system approximation.

The range of the time variation and the behavior of the predominant roots can be predicted to a large extent by a construction based on simplifying assumptions. This method makes the tracking of the influence of important stability derivatives very straightforward. With this method, considerable *a priori* knowledge can be provided on the behavior of the oscillatory roots during VTOL transition. This knowledge is used in a discussion of three different principles which can be used for artificial stability augmentation by means of feedback. The advantages of programmed feedback gain adjustment versus adaptive control are emphasized.

Recommendations.

The performance of an artificial stability augmentation system with programmed gain adjustments depends largely on the amount of information available. Therefore, further refinements of the presented approximating methods, especially in mid-transition, would be of great value. The full potential and limitations of such a system should be investigated by exploring the sensitivity of the performance to deviations from assumed design parameters.

INTRODUCTION

One common characteristic of low-speed flying machines is their ability to achieve rapid changes of the flight condition. The accent is on the word "rapid," and the present report is concerned primarily with this aspect of the flying characteristics of such vehicles.

The reason why low-speed flying machines exhibit this characteristic can be made plausible by considering Newton's law: $F = ma$. If a certain accelerating force, F, is acting on a body for a time interval, τ, the achieved velocity can be expressed in this form:

$$\frac{V}{V_o} = 1 + \frac{F}{mV_o}\tau$$

The smaller the initial velocity, V_o, the larger is the relative change in velocity. Most of the aerodynamic stability and control derivatives vary with V^2; therefore, a set of linearized differential equations can be expected to become invalid more rapidly at low speeds than at high speeds.

It should be noted here that the rapid changes of characteristics at very high speeds, although similar in effect, are of a different origin. The very high velocities enable the vehicle to encounter rapid changes of the environment; for example, of the air density.

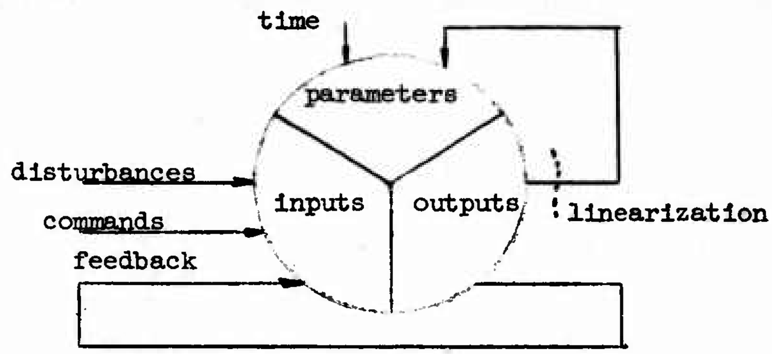
In both cases, at very low and at very high velocities, it is actually the nonlinear variation of the dynamic pressure which causes the conventional set of linear equations to become invalid. Therefore, the findings of this report, although presented with low-speed flying machines in mind, are applicable to dynamic problems at both extremes of the velocity range.

A rigorous approach to the outlined problem would lead to a set of nonlinear differential equations valid over the required velocity range. In the present state of the art such an approach could not be expected to yield useful results. A more promising approach tries to extend the classical method of small perturbations so that approximating results could be expected to reveal the essential behavior of the nonlinear system.

Consider a general physical system as described by a set of differential equations. There are three kinds of "elements" which can be distinguished in most differential equations:

1. Functions of the independent variables only (inputs).
2. Dependent variables and their derivatives (outputs).
3. Coefficients (parameters).

The equivalent terms generally used are in parentheses. A simplified representation of a fairly general system can be given in the following form:



This somewhat unconventional representation has been chosen in order to emphasize the general nature of the relationships. A nonlinear differential equation can be considered a variation of an approximating linear differential equation in which the coefficients are functions of the dependent variables. The line connecting the outputs with the parameters then represents the essential characteristic of nonlinear systems which is the dependence of coefficients on the dependent variables. Linearization means cutting this line as indicated, i.e., considering the coefficients to be independent of the dependent variables. Total removal of all lines affecting the parameters is equivalent to a further simplification to a constant parameter system described by differential equations with constant coefficients. This latter approach is used when small perturbations of an equilibrium (trim condition) are investigated. In this case there is a conventional linear system with or without feedback.

The effect of the loop, indicated by the line between outputs and parameters, on such a linearized system, will now be discussed. From the viewpoint of dynamic response, there are two distinct effects of the nonlinear relationships represented by the upper loop in the sketch.

1. Nonlinear effects may change the dynamics in the vicinity of the equilibrium so that linearized perturbation equations are not valid. For example, saturation, backlash, etc., cause such effects. Several methods (e.g., phase space, describing functions) can be used to deal with these problems.

2. Nonlinear effects may change the equilibrium (trim) conditions to such an extent that the coefficients of the linear perturbation equation must be changed. For example, the change of dynamic pressure or the change of mass with fuel consumption cause such effects. It is this type of effect with which the present report is concerned.

It must be emphasized that the above distinction is not based on mathematical but rather on practical considerations. Actually, the same nonlinearity may give rise to both types of effects, depending upon the forces acting on the system. The first case occurs if the dynamic motion resulting from disturbances and control actions involves the nonlinearity; however, the equations describing the dynamics remain unaltered throughout the regime. The problems arising from this situation are not of concern here. It is assumed throughout the present report that the "small perturbation equations" of the system are linear.

The second case occurs when an average disturbance or control action moves the equilibrium state along the nonlinearity. The change of the linear term in the Taylor series expansion of the nonlinearity means continuous changes in the coefficients of the small perturbation equations. This consideration leads to the following linearization of the nonlinear problem. It is assumed that the average forces and moments are continuously balanced (trimmed). Small deviations from the trim condition appear as forcing terms in the perturbation equations. If the assumption is now made that the time history of the trim conditions can be predicted, then the parameters of the perturbation equations can be considered functions of time instead of functions of the system output. With these assumptions the nonlinear system is approximated by a linear time varying system. The upper loop in the Figure on page 4 has been cut and the arrow pointing to the parameters represents a function of the time, t, which can be considered, in a generalized way, as an additional parametric input to the system.

If the predicted time variation is slow so that the parameters of the perturbation equation can be considered constant for the duration of at least one transient response, then the system can be further simplified to a "quasi-stationary" system. In this case, loci of the poles and zeros can be plotted with time as the running parameter. Such a plot represents the

so-called "frozen system" approximation to the time variable system. The somewhat vague fashion in which the "quasi-stationary" or "frozen" concept is introduced indicates the difficulty in defining a border line between time variable and constant parameter systems. Again, the distinction has to be based on practical engineering rather than on mathematical considerations. Obviously, problems arise when the time variation is such that the quasi-stationary approximation does not hold.

This report is concerned with the linear time variable approximation to the dynamic problems of low-speed flying machines. Only smooth and monotonic time variation are considered. Simplifications and approximations are applied widely in order to emphasize essential aspects rather than detailed calculations. One major aspect investigated is the effect of the speed of parameter variation. Various feedback stability augmentation configurations are considered. As an example, the transition of a VTOL aircraft is discussed in more detail.

THE FROZEN SYSTEM APPROXIMATION

Time varying linear systems can be described by linear differential equations in which the coefficients are functions of time. If the time variation is slow, the coefficients will not vary to any great extent within a given time interval. The definition of "slowness" involves the specification of the time interval to be considered as well as the specification of the coefficient variation to be tolerated. For practical reasons these specifications have to be determined with the actual output of the system in mind.

A stationary system is usually described either in the frequency domain (transfer function or amplitude and phase characteristic) or in the time domain (weighting function or impulse response). The frequency domain appears to be the less natural choice in which speed of time variation should be defined since it implies, by definition, steady state test frequency signals. The impulse response, on the other hand, provides the significant information within a limited time interval. A time interval, at the end of which it is possible to make a judgement on the essential characteristics of the system, seems useful as a basis for the investigation of the rate of time variation. The significance of the time variation can be judged by the deviation from the response of a stationary system. These ideas are now considered in more detail.

Assume a differential equation of a time varying system with an impulse applied at the time, τ, in the following form:

$$\sum_{i=o}^{n} f_i(t) \frac{d^i}{dt^i} x\,(t) = \delta(\tau) \tag{1}$$

If the variation of the coefficients, $f_i(t)$, is very slow, then the quasi-stationary equation

$$\sum_{i=o}^{n} f_i(\tau) \frac{d^i}{dt^i} y(t) = \delta(\tau) \tag{2}$$

is a good approximation. If the roots of the corresponding characteristic equation are plotted with τ as the running parameter, the frozen system loci of the time varying system are obtained. In general, the time variation of the zeros, as well as of the poles of a transfer function, can be

characterized in this way. The loci do not change if, τ, is replaced by, $\eta\tau$. The time variation is faster when η increases. Obviously, the approximation becomes worse with increasing η. The frozen solution, $y(t)$, can be used as a reference with which the solution, $x(t)$, of the time variable can be compared.

The problem is to define a time interval which permits a meaningful characterization of the frozen system solution and which can also be used for evaluating the deviation of, $x(t)$, from the frozen system solution, $y(t)$. Solving this problem in its full generality will not be attempted. Use will be made of two restrictions determined by the subject covered in this report: (1) to consider systems in which the time variation is smooth and monotonic, at least piecewise; (2) to examine the effect of the time variation from the point of view of its effect on a feedback control applied to the time varying system. The first restriction permits some highly simplifying approximations; the second restriction is of great help in determining a time interval for a meaningful characterization of the rate of time variation and its effects.

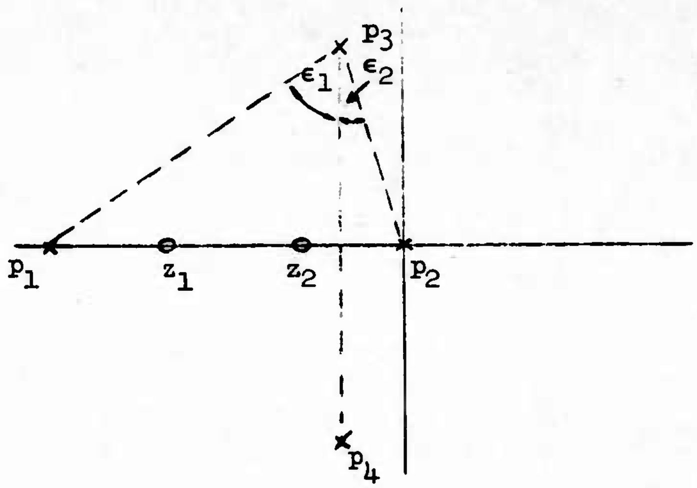
A discussion of some consequences of feedback control is presented first. The frozen system is considered then the deviations due to fast time variation. As mentioned before, the frozen system can be represented by loci of poles and zeros on the complex plane. At each instant, τ, determines a certain pole-zero configuration or pulse response. A pulse response of the frozen constant parameter system is the sum of component response modes. The modes are determined by the poles; the relative amplitudes and phases are determined by the relative location of the zeros and poles. Considering the response as the input to a feedback branch (automatic or human), it is important to distinguish between dominant modes and modes of minor significance. Even responses of systems of high order can usually be matched quite will with a second, third, or possibly fourth-order system response. Thus in the case of a time variable system the variation of the dominant modes is of primary significance.

It is very difficult to be quantitative about a general criterion for determining whether or not a mode should be considered predominant. However, this study is restricted to aircraft configurations. Equations of higher than fourth-order need not be considered. The relative importance of different modes should be judged on the basis of relative amplitudes.

The relative amplitudes of the different modes in the pulse response can be determined graphically from the pole-zero plot of the transfer function (Reference 1). The amplitude for any individual pole is determined by

dividing the product of the distances of the pole from the zeros by the product of the distances from the other poles. The amplitude of an oscillatory mode, represented by a pair of conjugate complex poles, is twice the amplitude determined for one of these poles. Any arbitrary scale can be chosen since only the relative amplitudes are of interest.

The following sketch shows some significant points in this procedure. Consider the following pole-zero configuration.



When the amplitudes of the oscillatory mode and the p_1 mode are compared in the form of a ratio, the distance between p_3 (or p_4) and p_1 drops out. The ratio can be written as

$$\left|\frac{A_{3,4}}{A_1}\right| = \frac{2\dfrac{|p_3 - z_1| \cdot |p_3 - z_2|}{|p_3 - p_4| \cdot |p_3 - p_2|}}{\dfrac{|p_1 - z_1| \cdot |p_1 - z_2|}{|p_1 - p_4| \cdot |p_1 - p_2|}} = \qquad (3)$$

$$= \frac{\dfrac{|p_3 - z_1|}{|p_1 - z_1|} \cdot \dfrac{|p_3 - z_2|}{|p_1 - z_2|}}{\dfrac{|p_3 - p_2|}{|p_1 - p_2|} \cos \epsilon_1}$$

In this example, z_2 is at a similar distance from both poles. The ratio of the distances to, p_2, together with cos ϵ_1, result in a factor of approximately two in favor of $A_{3,4}$. The decisive factor is caused by the fact that z_1 is much closer to p_1 than to p_3. Therefore, it can be safely estimated that the amplitude of the oscillatory mode is more than five times the amplitude of the p_1 mode.

The relative amplitude ratio A_2/A_1 of the p_1 and p_2 modes is determined in this example primarily by the square of the ratio of the distances to p_3 (or p_4) because the arrangement of the zeros is fairly symmetrical with respect to the two real poles. Therefore

$$\left|\frac{A_2}{A_1}\right| \approx \frac{\cos^2 \epsilon_2}{\cos^2 \epsilon_1} \qquad (4)$$

The considerations in this example can be utilized generally. For example, the conditions under which the short period mode of a pitch response is predominant can be easily estimated. Since no objective criterion for "predominant" can be established, some rule of thumb must be used. The amplitude of a predominant mode should not be less than four to five times the amplitude of any other mode. This is a reasonable criterion for a fourth order system.

NORMALIZATION

It is convenient to use a normalization so that the effect of time variation on a particular mode can be investigated in a general form. In order to investigate some fundamental aspects of the time variation, only one pair of complex conjugate roots is considered. This is equivalent to replacing, for example, an aircraft pitch attitude response with the short period oscillatory mode. This oversimplification will be helpful in the investigation of practical effects of fast time variation. The effect of deviations from this simplified model is discussed later.

The homogeneous differential equation of the simplified time varying system of second order is:

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x = 0 \qquad (5)$$

The solution in the vicinity of a frozen time solution is now examined. Using a conventional notation of fixed parameter systems

$$f_1(\tau) = 2\zeta_F\omega_{Fo}$$

$$f_2(\tau) = \omega_{Fo}^2$$

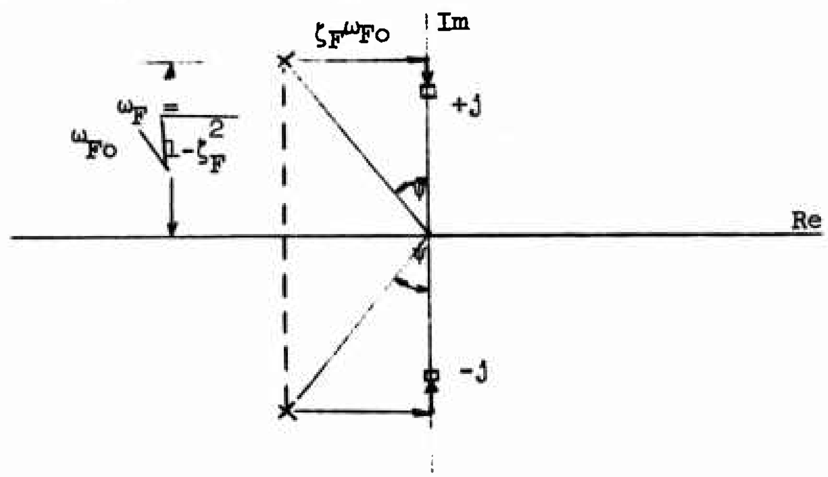
where the index, F, stands for "frozen."

In the course of transitions, the time variations of parameters are not only smooth but also monotonic over the full range, or at least over large portions of the range. Therefore, it is reasonable to investigate the effect of time variation by considering only the first two terms of the Taylor series of the coefficient time functions: the time variable damping and spring coefficients.

$$\ddot{x} + \left[2\zeta_F\omega_{Fo} + B(t - \tau)\right]\dot{x} + \left[\omega_{Fo}^2 + D(t - \tau)\right]x = \delta(\tau) \qquad (6)$$

For convenience, the origin of time is shifted to τ. Using the notations $B = \beta\omega_F^2$, $D = \mu\omega_F^3$, $\omega_F = \omega_{Fo}\sqrt{1 - \zeta_F^2}$

$$\ddot{x} + (2\zeta_F\omega_{Fo} + \beta\omega_F^2 t)\dot{x} + (\omega_{Fo}^2 + \mu\omega_F^3 t)x = \delta(o) \qquad (7)$$



Normalization is obtained by a linear transformation

$$y = xe^{\zeta_F\omega_{Fo}t}$$

and a change in time scale

$$t' = \omega_F t.$$

With

$$P = \beta$$

$$Q = \mu + \beta \tan \psi \tag{8}$$

the normalized equation is

$$\ddot{y} + Pt'\dot{y} + (1 + Qt')y = \delta(o) \tag{9}$$

Here t' is dimensionless time as defined above. This normalization puts the frozen poles into the (+j, -j) points of the S_y plane. The effect of time variation can be investigated in this plane. The simple transformation relationships are described in more detail in Appendix I.

The solution of the differential equation with linearly time varying coefficients can be obtained in the form of hypergeometric functions. However, for engineering purposes, such a solution in itself is not very helpful in the evaluation of fast time variation. An analog computer study was preferred because of the extreme flexibility offered by this approach. High accuracy is not important because of the approximations involved in the study. The goal is rather to identify significant effects of time variation.

ANALOG COMPUTER STUDY

The scheme for experiments with the normalized second order system was determined as follows. Since the normalization transforms straight lines into straight lines, it seemed advantageous to explore time variations which move the roots along straight lines, rather than purely linear time variations which move the poles along circles.

The normalized system equation

$$\ddot{y} + Pt'\dot{y} + (1 + Q_1t' + Q_2t'^2)y = 0 \tag{10}$$

determines the following locus for "frozen" roots:

$$S_y(t') = -\frac{Pt'}{2} \pm \sqrt{1 + Q_1t' + Q_2t'^2 - \frac{P^2t'^2}{4}} \tag{11}$$

If $Q_2 = \dfrac{P^2 + Q_1^2}{4}$, this locus becomes a pair of straight lines:

$$S_y(t') = -\frac{Pt'}{2} \pm j(1 + \frac{Q_1t'}{2}) \tag{12}$$

The slope $\tan^{-1}(-\frac{Q_1}{P})$ and the constant speed of variation

$$\left|\frac{dS_y}{dt'}\right| = \frac{P^2 + Q_1^2}{2}$$

are the same as those determined for the linearly time varying case at $t' = 0$ in Appendix I. It was decided that various speeds of variation along eight different directions 45° from each other, originating in the "frozen" pole pair $(+j, -j)$ should be investigated.

It was rather difficult to choose a basis for the evaluation of the effect of time variation. In the case of slow time variation the frozen system is a good approximation. The goal of the analog computer study was to explore the deviations from the frozen system solution due to fast time variation, from the viewpoint of the application of feedback. As stated above, in order to specify a speed of time variation a time interval and a criterion for the comparison must be chosen. The concepts of apparent damping and apparent frequency change were used in this choice. Although theoretically an "instantaneous" damping and frequency could be defined, these concepts

have no practical meaning because some finite time interval is necessary to measure damping and frequency. A human pilot is not very sensitive to slight frequency changes; damping can be judged most easily by peak ratios; therefore, at least one half cycle should be used for comparison. In order to provide a mechanical method for the comparison, the apparent damping and frequency were assumed to be the damping and frequency of an adjustable constant parameter system. The response of this system with an identical input was matched to the response of the time variable system. The match was made in the "least integral square error" (ISE) sense. This procedure was set up on an analog computer. It was found that using only one half cycle of the frozen system response for comparison was not selective enough; therefore, one full cycle of the undamped frozen system response was chosen as a comparison-interval.

A block diagram of the computer setup is shown in Figure 1. The equation of the normalized time variable system is given by

$$\ddot{y} + Pt\dot{y} + (1 + Q_1 t + Q_2 t^2)y = 0 \tag{13}$$

where Q_2 was chosen so that the roots moved along straight loci.

The constant coefficient system used to approximate the time varying system was

$$\ddot{y}_a + K_\zeta \dot{y}_a + (1 + K_\omega)y_a = 0 \tag{14}$$

where K_ζ and K_ω were adjustable parameters. Experiments at different speeds of variation were conducted by applying identical inputs to both systems and measuring the displacement error:

$$\epsilon(t) = y(t) - y_a(t)$$

This error was squared by a function generator and then integrated. The parameters K_ζ and K_ω were adjusted to find the particular combination which minimized the ISE over one period of the normalized frozen system response, 2π seconds.

$$ISE = \int_0^{2\pi} \epsilon^2 dt = \text{min} \tag{15}$$

Various speeds of variations were used for P and Q_1, varying from $1/4\pi$ to $2/\pi$ for increasing spring and/or damping coefficient; and from $1/8\pi$ for decreasing spring and/or damping coefficient. Obviously, the greater the speed, the larger is the ISE. Speeds less than $1/8\pi$ were barely noticed (K_ζ and K_ω are less than .05). Speeds greater than $1/\pi$ (300% of spring

constant variation) generated huge errors. Changing this time interval had little effect on K_ζ and K_ω for minimum ISE at slow speeds. At high speeds $(1/\pi, 2/\pi)$ K_ζ and K_ω were very sensitive to such changes. An example indicating the selectivity is shown in Figure 2.

Since very low speeds ($P, Q_1 \leq 1/8\pi$) were found to be approximated well by the frozen system solution and very high speeds can be omitted because of practical considerations, the following two speeds were selected for comparison:

$$P, Q_1 = 1/4\pi$$

$$P, Q_1 = 1/2\pi$$

The speed $Q_1 = 1/4\pi$ increased the spring constant by 50% in 2π seconds (the period of the frozen system). The speed of $Q_1 = 1/2\pi$ doubles the spring constant in 2π seconds. The speeds $P = 1/4\pi$ and $P = 1/2\pi$ mean that, if $Q_1 = 0$, a frozen system damping factor of .5 and .707, respectively, is reached after 2π seconds.

Figures 3 and 4 show the apparent roots at these speeds in different directions. The tips of the vectors indicate the frozen system roots at 2π seconds. Figure 5 combines these two figures by normalizing to the vector lengths.

These data, combined with data obtained with other speeds, are plotted in a different way in Figure 6. The heavy line in this figure indicates the boundary between those values of P and Q_1 which result in $K_\zeta > 0$ and those which result in $K_\zeta < 0$. For the range of speeds of condern it can be seen that the apparent "destabilizing" effect of negative P can be cancelled by a positive Q of twice the magnitude. For large values of negative Q, the dynamic effect will be a decrease of apparent damping no matter how large P. (This is meaningful only for $Q_1 > -1/2\pi$ since if $Q_1 < -1/2\pi$, the roots become real within 2π seconds).

In order to estimate the effect of fast parameter variation for oscillatory roots anywhere in the complex plane, the inverse of the normalizing transformation must be performed. The apparent damping found in the normalized form estimates the change in the damping of the frozen system output. This means that the larger the frozen damping coefficient, the less the apparent deviation caused by a fast time variation. Therefore, it can be stated that the region of interest for this effect is that of low positive andnegative damping. In those phases of a VTOL transition where the oscillatory roots are in the vicinity of the imaginary axis, it can be assumed that

deviations from the frozen system assumption will influence the handling qualities and the performance of autopilots designed on a frozen system basis if the parameter variation is fast. It has been shown that fast variation of the spring constant means a 50% to 100% change within the time of one frozen system period. A decreasing spring constant causes more deviation from the frozen system than an increasing spring constant. Fast variation in damping means a variation of .5 to .7 in the damping ratio during one frozen system period. If a combination of spring and damping variation takes place, the effects may strengthen (or weaken) each other with the result that a smaller (or larger) speed of variation of the individual parameters can cause a similar effect on the apparent damping.

For VTOL transitions the significance of these findings is that the effects caused by fast parameter variations are expected to be stronger when frequency and damping are decreasing rapidly in the vicinity of the imaginary axis than when the roots move in the opposite direction along a frozen root locus.

APPROXIMATIONS TO THE LONGITUDINAL DYNAMICS OF A VTOL AIRCRAFT

The simplified linearized rigid-body equations which describe the perturbed longitudinal airplane motion about a straight and level flight path may be written as follows:

$$(S - X_u) + X_w w + g\theta = 0 \qquad (16)$$

$$- Z_u u + (S - Z_w)w - VS\theta = 0$$

$$- M_u u - M_w w + (S - M_{\dot{\theta}})S\theta = M_\delta \delta$$

u, w, θ, are the small variations of the perturbed flight variables around the equilibrium point. The coordinate system is fixed to the body. The X-axis is taken positive forward and is initially set horizontal. The Z-axis is positive downward. The only input considered in these equations is the control moment around the pitch axis of the airplane.

1. NEAR-HOVERING FLIGHT

The description of the longitudinal dynamics in near-hovering flight can be simplified by the assumption that the vertical degree of freedom is uncoupled from the other two (the horizontal and the angular) degrees of freedom. This is justified as long as the stability derivatives X_w (variation of horizontal force with vertical velocity) and M_w (variation of pitch moment with vertical velocity) are small.

These assumptions are commonly used for satisfactorily approximating the modes of motion of helicopters near hovering. For VTOL machines M_w can be expected to remain negligible for a larger velocity range near hovering than is the case for conventional single rotor helicopters, because the destabilizing contribution by the rotors which increases as the VTOL airplane gains speed is balanced to some extent by the tail effect. Moreover, for VTOL machines of the tilt-wing type, the effect of X_w on the transfer function is expected to be small since the rotor and wing contributions to this derivative are of opposite signs. These considerations encourage us to consider the assumptions made for the near-hovering regime to be valid over a considerable range at low speeds.

The transfer function for the attitude response within this range can be expressed as follows:

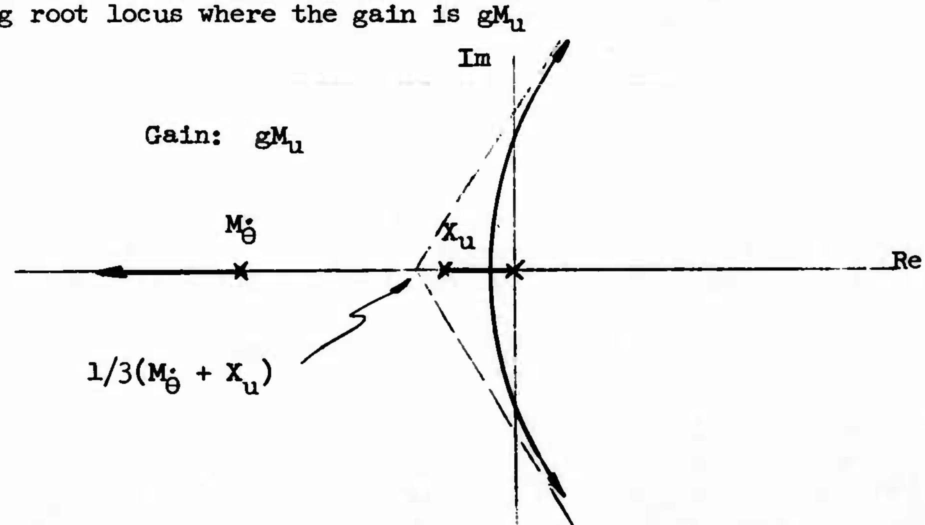
$$\frac{\theta}{\delta} = \frac{\begin{vmatrix} S - X_u & 0 & 0 \\ -Z_u & S - Z_w & 0 \\ -M_u & 0 & M_\delta \end{vmatrix}}{\begin{vmatrix} S - X_u & 0 & g \\ -Z_u & S - Z_w & -SV \\ -M_u & 0 & S(S - M_{\dot{\theta}}) \end{vmatrix}} \tag{17}$$

$$= M_\delta \frac{(S - Z_w)(S - X_u)}{(S - Z_w)\left[S(S - M_{\dot{\theta}})(S - X_u) + gM_u\right]}$$

Hence, the Z_w mode is cancelled as far as the θ response is concerned (this mode is also cancelled for the u response). This leads to the following third order characteristic equation for low speeds:

$$S(S - M_{\dot{\theta}})(S - X_u) + gM_u = 0 \tag{18}$$

The location of the characteristic roots can be determined from the following root locus where the gain is gM_u



The larger gM_u, the closer are the oscillatory roots to the asymptotes which originate in $1/3(M_\theta + X_u)$. For large enough gM_u the three original poles ($M_{\dot\theta}$, X_u, and zero) can in effect be replaced by a triple pole at $1/3(M_{\dot\theta} + X_u)$. In this case

$$\left[S^* - 1/3(M_{\dot\theta} + X_u)\right]^3 + gM_u = 0 \tag{19}$$

can be considered as an approximating characteristic equation. This approximation places the poles right on the asymptotes at equal distances of $\sqrt[3]{gM_u}$ from the $1/3(M_{\dot\theta} + X_u)$ point. If the actual roots are S_1, S_2, S_3, and if the approximating roots are S_1^*, S_2^*, S_3^*, the error of approximation can be expressed as

$$\begin{aligned} (S_1^* - S_1) &= 2\epsilon_x \\ (S_2^* - S_2) &= -(\epsilon_x + j\epsilon_y) \\ (S_3^* - S_3) &= -(\epsilon_x - j\epsilon_y) \end{aligned} \tag{20}$$

Using the results obtained for ϵ_x, ϵ_y in Appendix II, it can be shown that

$$\epsilon_x \approx \frac{1}{6}\,\frac{1/3(M_{\dot\theta} + X_u)^2 - M_{\dot\theta}X_u}{\sqrt[3]{gM_u}} \tag{21}$$

$$\epsilon_y \approx \sqrt{3}\,\epsilon_x \tag{22}$$

For a VTOL with a reasonable amount of static stability with velocity, M_u, ($gM_u \sim 1/\sec^{-3}$) and expected values for $M_{\dot\theta}$ and X_u, we find $\epsilon_x < .02\ \sec^{-1}$. Therefore, for practical purposes it is sufficient to consider only the variations of the parameters $1/3(M_{\dot\theta} + X_u)$ and $\sqrt[3]{gM_u}$ as affecting the displacement of the roots.

$$S_i = (S_i)_{hov} + 1/3\,\Delta(M_\theta + X_u) + \Delta\sqrt[3]{gM_u}\,e^{j(60^\circ + i\,120^\circ)} \tag{23}$$

$$i = 1,\ 2,\ 3$$

This means that the roots will move horizontally to the left for an increasing damping and radially towards the c.g. of the three roots for a decreasing static stability with velocity.

The maximum speed to which this approximation can be used depends primarily upon the increase of M_α ($=VM_w$) with velocity. Experimental results with different VTOL planes (References 1, 2, 3) show that first M_α increases with velocity, because of the inherent rotor instability with angle of attack. Shortly thereafter, the tail stabilizing contribution overrides that effect and M_α decreases and becomes negative. As a result, there is a velocity at which the cubic characteristic equation accurately represents the dynamics of the vehicle. It is therefore reasonable to consider our approximation as accurate enough for the interval from hovering up to this point.

2. NEAR-FORWARD FLIGHT

As the airplane approaches its forward flight configuration, the static stability with angle of attack increases considerably. The transfer function for the pitch attitude becomes

$$\frac{\theta}{\delta} = \frac{\begin{vmatrix} S - X_u & X_w & 0 \\ -Z_u & S - Z_w & 0 \\ -M_u & -M_w & M_\delta \end{vmatrix}}{\begin{vmatrix} S - S_u & X_w & g \\ -Z_u & S - Z_w & -SV \\ -M_u & -M_w & S(S - M_{\dot\theta}) \end{vmatrix}} = \qquad (24)$$

$$= M_\delta \frac{(S - Z_w)(S - X_u) + X_w Z_u}{(S - Z_w)\left[S(S - M_{\dot\theta})(S - X_u) + gM_u\right] - M_w V(S^2 - X_u S - g/V\, Z_u) + X_w S\left[Z_u(S - M_{\dot\theta}) - VM_u\right]}$$

The derivative X_w is not negligible now in comparison with some other derivatives; yet the X_w terms in both numerator and denominator can still be disregarded on the basis that at higher speeds M_u is small and Z_w is much larger than in hovering. In other words, it is assumed

$$X_w Z_u << Z_w X_u$$

$$X_w M_u << M_w X_u$$

Hence

$$\frac{\theta}{\delta} = M_\delta \frac{(S - Z_w)(S - X_u)}{(S - Z_w)\left[S(S - M_{\dot\theta})(S - X_u) + gM_u\right] - M_\alpha(S^2 - X_u S - g/V\, Z_u)} \qquad (25)$$

It is a common practice to study the short-period roots of conventional airplanes using a second order equation derived from the fourth order characteristic equation when the higher frequency roots are far away from the other two roots. By the same reasoning, the same procedure can be used in dealing with VTOLs during the last part of the transition; but it should be remembered that tha farther the flight condition is from forward flight, the poorer will be this approximation.

In order to show the influence of various parameters on the approximation, the root locus technique is used to determine the characteristic roots. (Reference 4). First, the low-frequency roots ("phugoid" roots) are considered.

From the expression

$$S(S - M_{\dot\theta})(S - X_u) + gM_u \qquad (26)$$

which appears in square brackets in the denominator of the transfer function θ/δ (equation25) one can determine three points on the complex plane. The roots of the equation are on a root locus as sketched in Figure 7A with the gain gM_u.

M_u is assumed to be small and positive throughout the range near full speed. Two of the roots will then be either on the real axis within the interval $(0, X_u)$ or close to the real axis on the curved branch, as shown in the sketch. Next, another root locus can be used to obtain the roots of the characteristic equation. The starting poles for this locus are the three poles obtained as roots of the equation (26), and a pole at Z_w. The zeros are the roots of the equation:

$$S^2 - X_u S - g/V\, Z_u = 0 \qquad (27)$$

and the gain is $- M_\alpha$. This locus is shown in Figure 7B.

The zeros are most likely a complex pair close to the real axis. They lie on the vertical line passing through the point ($1/2\ X_u$, $0j$) and their distance from the origin is $(g/V\ Z_u)^{1/2}$. The effect of a large gain, $-M_\alpha$, is to bring the low-frequency roots of the characteristic equation very close to the zeros given by equation (27). The result is the well-known approximation of the phugoid roots.

The above presentation shows how the approximation deteriorates for smaller M_α. In this case the approximation overestimates the frequency of the phugoid roots. Consequently, $(g/V\ Z_u)^{1/2}$ is a limit for the frequency of the phugoid roots. The influence of all the other derivatives on these roots can be easily followed by estimating changes in the geometry of Figure 7.

The two modes of motion, the phugoid and the short-period mode, are well separated during the high-speed phase of the transition. Since the phugoid roots are considerably closer to the zeros of the transfer function θ/δ, the short-period mode can be considered to be the predominant mode of the pitch-attitude response.

An analysis follows of the second order approximation of the characteristic equation

$$\lambda^2 - (M_{\dot\theta} + Z_w + X_u)\lambda + (Z_w + X_u)M_{\dot\theta} - M_\alpha + Z_w X_u = 0 \qquad (28)$$

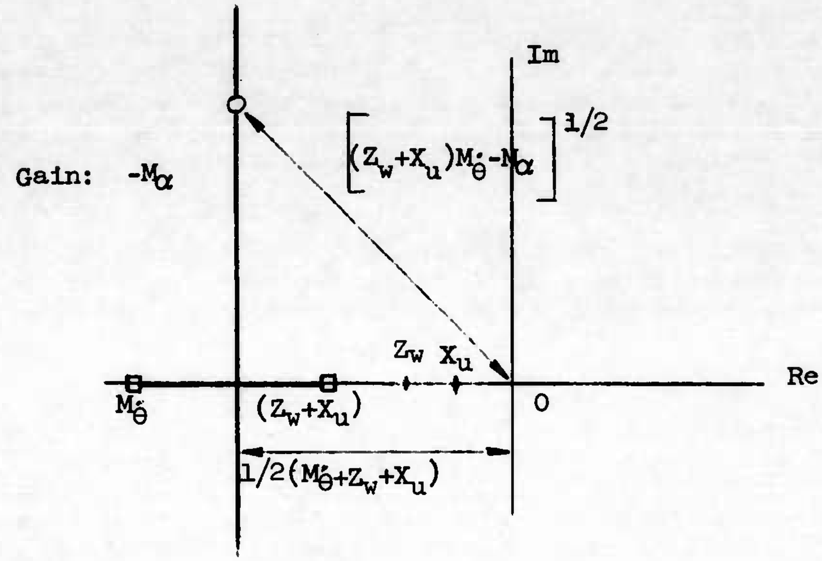
For VTOL configurations with improved dynamic and aerodynamic characteristics the following inequality can be expected to be valid during the high-speed portion of the transition:

$$M_{\dot\theta} \gg X_u$$

This leads to further simplification of the second order equation:

$$\lambda^2 - (M_{\dot\theta} + Z_w + X_u)\lambda + (Z_w + X_u)M_{\dot\theta} - M_\alpha = 0 \qquad (29)$$

Using the root locus technique is again helpful in tracing the effects of the derivatives. The starting poles of the root locus



are $M_{\dot\theta}$ and $Z_w + X_u$. The gain determining the roots is $- M_\alpha$. The roots lie on a vertical line perpendicular to the real axis, crossing the axis at the arithmetic mean of $(Z_w + X_u)$ and $M_{\dot\theta}$.

The constant term in equation (29), $(Z_w + X_u)M_{\dot\theta} - M_\alpha$, represents the square of the distance between either root and the origin. If the difference between $M_{\dot\theta}$ and $(Z_w + X_u)$ is considerably smaller than their absolute values then their product which appears in the constant term can be substituted by the square of their arithmetic mean. (The error introduced by this assumption is only 6% if their difference is half their arithmetic mean value) With this substitution it can be shown that $(- M_\alpha)^{1/2}$ is a good approximation to the distance of the roots from the real axis. The real part of the short-period roots is approximated, as stated before, by $1/2(M_{\dot\theta} + Z_w + X_u)$. The damping ratio depends on the parameter $(M_{\dot\theta} + Z_w + X_u)/2(- M_\alpha)^{1/2}$.

3. CONSTRUCTION OF TRANSITION LOCI NEAR HOVERING AND NEAR THE END OF THE TRANSITION.

The approximations of the oscillatory roots near hovering and near the end of the transition can be obtained graphically as shown in Figure 8.

According to the previous approximations two lumped parameters in terms of stability derivatives are introduced for the near-hovering regime ($\sqrt[3]{gM_u}$ and $1/3\left[M_{\dot\theta} + x_u\right]$) and two others for the regime near the forward flight end of the transition ($\sqrt{-M_\alpha}$ and $1/2\left[M_\theta + Z_w + X_u\right]$). The values of these parameters can be assumed to be known for hovering and forward flight (completed transition).

The changes in these parameters determine the approximate changes in the oscillatory roots near both end points of a transition. Figure 8 illustrates boundaries for assumed changes of the parameters with respect to the nondimensional velocity $V/V_{forward\ flight}$, and their effect on variations of the oscillatory roots.

The assumed boundaries are shown in auxiliary coordinate systems by means of two lines for each parameter. Parameter variations described by any line running between these limit lines will cause root variations within the constructed resultant boundaries on the S-plane. As an example, an upper boundary line for the location of the short-period roots is obtained by combining the line of assumed minima of the damping, $1/2(M_{\dot\theta} + Z_w + X_u)$, for the near-forward flight situation with the line of assumed maxima of the frequency $\sqrt{-M_\alpha}$.

In the near-hovering flight region the roots are determined graphically by their coordinates in an auxiliary axis system in which the angle between the axes is 60°. The root coordinates in this system are given by $\sqrt[3]{gM_u}$ and $1/3(M_{\dot\theta} + X_u)$. Similarly, the upper and lower boundaries of the roots are determined as above.

The plot of the roots on the S-plane shows a gap between the lines <u>a a'</u> and <u>b b'</u> on which the roots corresponding to 0.4 $V/V_{f.f.}$ are located when obtained by the near-hovering and the near-forward flight approximation. This is the natural consequence of the inaccuracies in the approximations. The actual size of the gap also depends on the relative location of the hovering and the forward flight roots.

4. MID-TRANSITION FLIGHT

The mid-transition flight regime can be defined as that range where M_α is too small for the assumption made in the near-forward-flight approximation, but cannot be neglected as was done in the near-hovering flight assumptions. Moreover, Z_w, which depends upon the wing operating conditions, can change largely within this range whenever the wing operates in the nonlinear region

of the lift curve with respect to effective angle of attack. Simplifying approximations like those made for both end regions of the transition result in a grossly distorted picture, because it is characteristic for the mid-transition regime that all the important stability derivatives are of comparable significance. Despite this fact, the construction of Figure 8 is extended into the mid-transition region. It is interesting to note that even this oversimplification gives an indication of the general behavior of the oscillatory roots throughout the transition. In particular, both an early increase in Z_w and a fast decrease of M_u, with increasing velocity, indicate a "dip" of the oscillatory roots toward the real axis. However, it cannot be predicted by the illustrated procedure how far this dip actually extends in mid-transition flight. In fact, two alternatives for the general behavior of the roots can be recognized. It is possible that the hovering roots move (perhaps through a "dip") toward the short-period roots and that the two real roots in hovering combine into the phugoid roots in forward flight (Figure 9A). The alternative possibility is that the hovering roots move toward the phugoid roots during the transition and that the real roots in hovering combine and break away toward the short-period roots somewhare in mid-transition (Figure 9B). The loci in Figure 9 are not root loci in the usual sense but describe the motion of the characteristic roots in the complex plane during transition.

Further application of the root locus technique indicates that these two different behaviors for the roots depend primarily on the relationship between Z_w and M_α during mid-transition. If M_α increases much earlier in the course of the transition than Z_w, the locus of Figure 9A will be the result. If Z_w increases earlier than M_α, the locus will be of the type illustrated in Figure 9B.

In general, the variation of M_u , $M_{\dot\theta}$, Z_w , and M_α in mid-transition determines the eventual amount of "dip" in the movement of the oscillatory roots. A sharp dip implies a large decrease and increase within a relatively small speed range. In such a case the results of the previous section on the effects of fast time variation apply.

Apparent deviations from the frozen system approximation can be expected to influence the handling qualities and feedback control. It has been shown that a fast decrease in frequency near the imaginary axis can cause a considerable decrease in apparent damping. The approximate analysis presented in this section has identified the possible causes leading to this situation. The method can be used in a quick evaluation of the effects of design changes on the behavior of the oscillatory characteristic roots during a transition.

5. OSCILLATORY ROOTS IN TRANSITION

In order to illustrate transitions of the simplified second order model described earlier, the piecewise straight transition locus shown in Figure 10A has been chosen. The component of speed of variation parallel to the real axis was kept constant. Pulse inputs were applied automatically at the instants when the roots were in the positions marked 1, 2, 3, 4; only one point was tested in each transition. Transients for the fairly fast total transition time of 32 seconds are illustrated in Figures 10B through 10E. This time is approximately five times the period of the hovering or mid-transition response. The frozen system pulse response (a) and the responses for the accelerating (b) and decelerating (c) transition are shown for comparison in each figure. Both the displacement and the velocity responses are shown.

Transients for an even faster transition of 16 seconds (approximately 2.5 periods) are shown superimposed in Figures 11A to 11D for the points illustrated in Figure 10A. Note the considerable destabilizing effect during deceleration and the stabilizing effect during acceleration.

FEEDBACK CONSIDERATIONS

In this final section of the report some aspects of artificial pitch-attitude stability augmentation of VTOL aircraft are discussed. These aspects are quite general in their nature and part of the discussion applies to all time varying feedback control systems in general. Some important problems are omitted, such as the provision of sufficient control authority for the feedback system. Emphasis is placed on those aspects which are connected with the time varying nature of the problem. The range of variations can be estimated, based on the frozen approximation. The deviation from the frozen can be estimated, based on the results presented previously in this report.

In order to provide a basis for the choice of the feedback configurations, the goal to be achieved must first be determined. Since the concern is with piloted aircraft, some variation in frequency as well as in damping of the predominant closed loop response can be tolerated because of pilot adaptivity. A range of low frequencies within a 2:1 ratio and a range of relative damping ratios between .2 and .7 can be handled easily by a pilot Such tolerances define a considerable area in the complex plane within which the predominant closed loop roots may possibly be located.

We assume a basic rate and attitude feedback loop in which the feedback gains have to be adjusted because of the time variation of the system. The following alternatives will be considered as fundamentally differing feedback adjustments for time varying feedback systems.

1. ADJUSTMENT BY DIRECT FEEDBACK

In principle, it would seem desirable to make adjustments in the basic feedback loop according to the variation of the variable which is the physical cause of the time variation. This is the dynamic pressure or velocity for VTOLs and helicopters, and a more complicated pressure distribution in the case of ground effect machines. If the time varying system is well known, then the proper gain adjustments as functions of the velocity can be predetermined. The instrumentation of such a scheme involves the measurement of the velocity, some nonlinear device determining the amount of gain adjustment and the gain adjusting element itself (Figure 12A). Several difficulties make this approach unfeasible for practical application to low-speed flying machines. Dynamic pressures cannot be measured well enough at very low speeds. However, even if they could be determined adequately, two other fundamental problems remain. In the frozen system approach, the uncertainty about the actual variations of derivatives makes the determination

of the nonlinear adjustment functions somewhat arbitrary. Additional conditions such as the amount of acceleration or deceleration at a certain velocity cause large variations and additional comlexity.

The adjustment by direct feedback means an additional feedback loop around the system with the basic loop. The two loops are coupled through the gain adjustment. In the case of slow time variation, the variable on which the adjustment depends, the velocity, can be filtered in order to extract the average variation from the time history of the velocity. For fast time variations, however, when significant changes take place within one period of a predominant mode, the filtering time constant has to be so short that dynamic coupling between the two loops cannot be avoided. This causes an additional problem because the dynamics of the entire system become increasingly complicated. Actually, the linearization which leads to a time varying linear system is not valid in this case. For these reasons gain adjustments by means of direct feedback should not be used for the purpose in mind here.

2. ADAPTIVE FEEDBACK

The principle of adaptive feedback was introduced to handle control problems where the system to be controlled is not sufficiently well defined for a conventional feedback system synthesis. The term "adaptive" is used here for systems in which the performance of the feedback control is determined in terms of a performance index, and adjustments are made accordingly in the basic feedback loop (Figure 12B). Based on the frozen system approximation, two major difficulties which were connected with the first approach could be eliminated by an adaptive control system but at the expense of added complexity. These are the problems of velocity measurement at low speeds and the effects of design uncertainties. However, the less the time available for system identification, the larger will be the instantaneous deviations of the gain settings from the optimum. If an adaptive system is expected to follow changes within one period of a predominant mode, then dynamic coupling between the adaptive loop and the basic feedback loop cannot be avoided. Fast system identification is complicated even theoretically and could lead to very complex instrumentation.

One way of getting around this problem is to suppress the original predominant mode of the system. This implies that a considerably higher feedback gain can be applied than would be necessary for stability augmentation itself, so that the natural frequency of the predominant closed loop mode is increased to be fast with respect to the time variation. The result is a very tight control system implying that more control authority must

be available than would be necessary for stability augmentation only. For example, in the Minneapolis-Honeywell adaptive autopilot such a tight loop around the aircraft is used to make the aircraft follow a dynamic model response.

Whenever the application of adaptive control is considered, as much *a priori* knowledge as possible should be used in order to make the design as simple and as efficient as possible. It has been shown in the section, "Approximations to the Longitudinal Dynamics of a VTOL Aircraft," that even a very simplified description leads to considerable information on the behavior of the predominant roots. Determining the roots in the forward flight configuration and hovering are classical problems. An area inside which the oscillatory roots move in transition can be reasonably well approximated; the actual root will not be too far from the crude approximation. A question must be raised as to whether the complexity of an adaptive system is justified or whether a much simpler approach might not be as effective.

3. PROGRAMMED ADJUSTMENTS

The considerable amount of *a priori* knowledge and the rather broad tolerance permitted because of the pilot's adaptivity suggest the feasibility of simple feedback gain adjustments. The range of open loop damping is determined by the hovering and the forward flight roots. The frequency range is influenced largely by the "dip" of the transition locus (Figure 8). The simplifying approximations made throughout this report can serve as a basis for programming the feedback adjustments. Programming means that the adjustments are "open loop" adjustments rather than being derived from the output of the system as in the previous two configurations (Figure 12C). Adjustments are programmed as functions of the wing angle in the Tri-Service VTOL aircraft. The closed loop roots of an approximating model can be confined to a small region of the complex plane by means of rather simple gain programs. The neglected poles and zeros, variations of the c.g. of the aircraft, and deviations due to acceleration and deceleration will cause the closed loop roots to move within a larger region than the one predicted by neglecting these effects. Thus it is necessary to rely on the adaptivity of the pilot. The effect of fast time variation must also be taken into account, using the relationships established in the Section, "Analog Computer Study" and Appendix I. It has been shown that the apparent effects due to fast time variation are strongest near the imaginary axis. Therefore, with increased closed loop damping the effect of the fast time variation becomes less apparent.

The simplicity of programmed feedback adjustments is very attractive. Further investigation of the sensitivity of such a system to variations and uncertainties of system parameters is under way in order to establish the full potential and limitations of this approach. Whenever the performance of this approach can be made acceptable it should be preferred to the more elegant but much more complex adaptive approach.

Similar conclusions were reached in a previous report (Reference 5) in connection with the artificial stability augmentation of ground effect machines.

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APPENDIX I

NORMALIZATION

The normalization of

$$\ddot{x} + (2\zeta_F\omega_{Fo} + \beta\omega_F^2 t)\dot{x} + (\omega_{Fo}^2 + \mu\omega_F^3 t) = 0 \qquad \text{(A1)}$$

through $y = xe^{\zeta_F\omega_{Fo}t}$ and $t' = \omega_F t$ where $\omega_F = \omega_{Fo}\sqrt{1 - \zeta_F^2}$

leads to

$$\ddot{y} + Pt'y + (1 + Qt')y = 0 \qquad \text{(A2)}$$

This transformation is equivalent to a horizontal shift of the frozen poles by $\zeta_F\omega_{Fo}$ into the imaginary axis and a scale expansion on both axes by the factor $1/\omega_F$. Lines passing through the frozen point on the S_x plane are transformed into lines passing through the $(+j, -j)$ point on the S_y plane. The transformation is angle preserving. The linear time variation of the coefficients causes the frozen roots to move along circles centered on the real axis. These circles go over into circles in the course of the transformation. The relationships between P, Q and β, μ was given, by definition

$$P = \beta; \quad Q = \mu + \beta \tan\psi \; \left(\psi = \tan^{-1} \frac{-\zeta_F}{\sqrt{1 - \zeta_F^2}}\right) \qquad \text{(A3)}$$

The slope of the variation as well as the rate of variation in the normalized S_y plane can be determined as functions of β and μ. The normalized frozen locus is obtained by taking t' as a parameter in the following characteristic equation.

$$S_y^2 + Pt'S_y + (1 + Qt') = 0 \qquad \text{(A4)}$$

$$S_y(t') = 1/2 \left[- Pt' \pm \sqrt{(Pt')^2 - 4(1 + Qt')}\right] \qquad \text{(A5)}$$

For the variation of S_y in the vicinity of $t' = 0$

$$\left.\frac{dS_y(t')}{dt'}\right|_{t'=0} = 1/2(P \pm jQ) \qquad \text{(A6)}$$

The slope of the locus is determined by

$$\angle \left. \frac{dS_y(t')}{dt'} \right|_{t'=0} = \gamma = \tan^{-1} \frac{Q}{P} = \tan^{-1} \left(\frac{\mu}{\beta} - \frac{\zeta_F}{\sqrt{1 - \zeta_F^2}} \right) \qquad \text{(A7)}$$

$-\frac{\zeta_F}{\sqrt{1 - \zeta_F^2}} = \tan \psi$ where ψ is the angle enclosed by the radius vector of the pole and the imaginary axis. μ/β can be considered the tangent of a fictitious angle ϕ. This permits the illustration of the full range of β and μ, including $\beta = 0$, on a finite scale. The angle γ of the normalized frozen root variation is related to μ/β and ζ_F of the original root through the following relationship

$$\tan \gamma = \tan \phi + \tan \psi \quad \text{where} \quad \phi = \tan^{-1} \frac{\gamma}{\beta} \qquad \text{(A8)}$$

This relationship is plotted with ζ_F as a parameter on Figure 13. The speed of root variation is also determined by β, μ and ζ_F.

$$\left| \left. \frac{dS_y(t')}{dt'} \right|_{t'=0} \right| = 1/2 \sqrt{\beta^2 + \left(\mu - \frac{\zeta_F}{\sqrt{1 - \zeta_F^2}} \beta\right)^2} \qquad \text{(A9)}$$

For poles along the imaginary axis, where $\zeta_F = 0$, this becomes

$$1/2 \sqrt{\beta^2 + \mu^2}$$

The general case can be related to this value.

$$\left| \left. \frac{dS_y(t')}{dt'} \right|_{t'=0} \right| = 1/2 \sqrt{\beta^2 + \mu^2} \; \frac{\sqrt{\beta^2 + \left(\mu - \frac{\zeta_F}{\sqrt{1 - \zeta_F^2}} \beta\right)^2}}{\sqrt{\beta^2 + \mu^2}} \qquad \text{(A10)}$$

$$= 1/2 \sqrt{\beta^2 + \mu^2} \; \frac{\sqrt{1 + (\tan \phi + \tan \psi)^2}}{\sqrt{1 + (\tan \phi)^2}}$$

The coefficient of $1/2 \sqrt{\beta^2 + \mu^2}$ with ζ_F as a parameter, is illustrated in Figure 14.

The shapes of the transformed frozen pole loci with time variation are circles with their centers on the real axis at the distance

$$Q/P = \frac{\mu}{\beta} - \frac{\zeta_F}{\sqrt{1 - \zeta_F^2}} = \tan \Phi + \tan \psi$$

and the radius equal to

$$\sqrt{1 + (\tan \phi + \tan \psi)^2}$$

APPENDIX II

NEAR-HOVERING-FLIGHT APPROXIMATION

We can determine the error made in the determination of roots when the near-hovering characteristic equation

$$S(S - M_{\dot{\theta}})(S - X_u) + gM_u = 0 \tag{A11}$$

is substituted by the following equation

$$\left[S^* - 1/3(M_{\dot{\theta}} + X_u)\right]^3 + gM_u = 0 \tag{A12}$$

For convenience we shift the origin of the coordinate system into the c.g. of the roots of both equations into the $(1/3M_\theta + X_u), 0)$ point. This transformation does not change the errors nor the c.g. of the roots.

The new cubic equation obtained from equation (A11) is of the type

$$\lambda^3 + B\lambda + C^3 = 0 \tag{A13}$$

where

$$B = - \left[1/3(M_\theta + X_u)\right]^2 \cdot \left[3 - \frac{M_{\dot{\theta}}X_u}{\left[1/3(M_\theta + X_u)\right]^2}\right]$$

$$C^3 = gM_u - \left[1/3(M_\theta + X_u)\right]^3 \cdot \left[2 - \frac{M_\theta X_u}{\left[1/3(M_\theta + X_u)\right]^2}\right]$$

Since M_u is large for low speeds

$$gM_u \gg \left[1/3(M_\theta + X_u)\right]^3 \cdot \left[2 - \frac{M_\theta X_u}{\left[1/3(M_\theta + X_u)\right]^2}\right]$$

and

$$C \simeq (gM_u)^{1/3}$$

It can also be shown that $B < C$

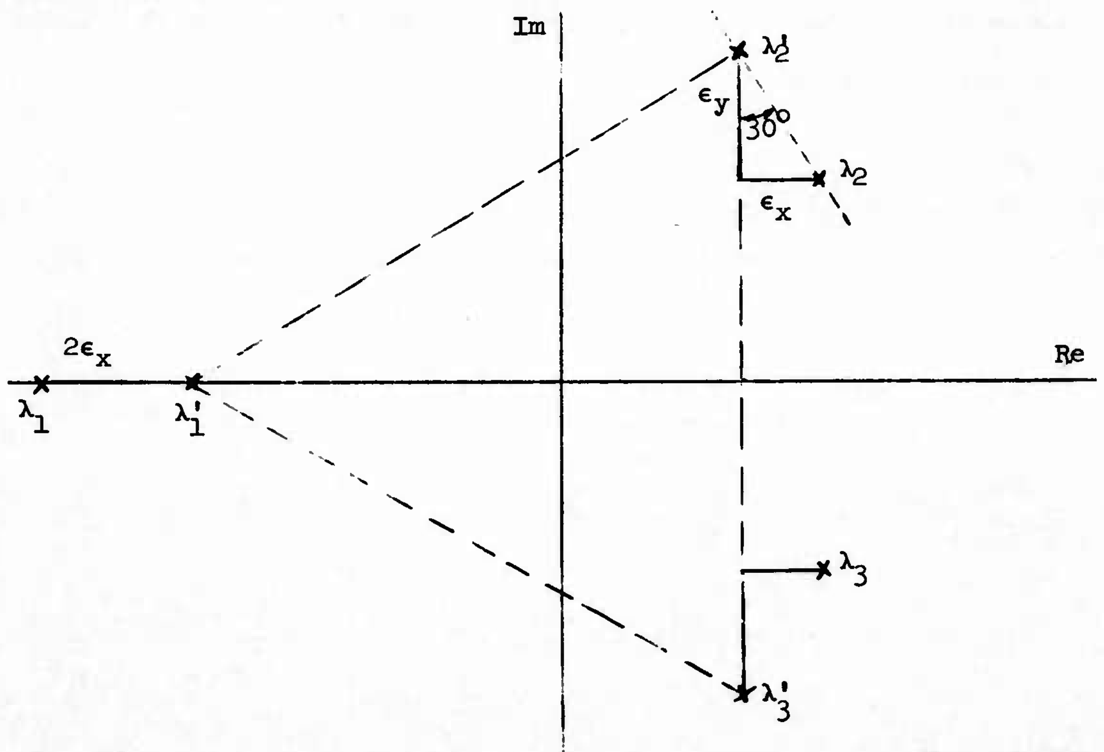
We solve the approximating equation which is obtained from equation (A12) as

$$\lambda'^3 + C^3 = 0 \qquad \text{(A14)}$$

then add to the roots correcting terms which are explicit functions of the coefficients B and C. The roots of equation (A14) are

$$\lambda_1' = C, \qquad \lambda_{2,3}' = 1/2C \pm j1/2\sqrt{3}\ C \qquad \text{(A15)}$$

These roots form an equilateral triangle in the complex plane. The roots of equation (A13) differ from these roots. The components ϵ_x and ϵ_y of the error are indicated in the following figure



The roots of the exact equation can be expressed as

$$\lambda_1 = - C - 2\epsilon_x , \qquad \lambda_{2,3} = 1/2C + \epsilon_x \pm j(1/2 \sqrt{C} - \epsilon_x) \qquad \text{(A16)}$$

The coefficient B is equal to the sum of all double products of the roots

$$B = (- C - 2\epsilon_x)2\ (1/2C + \epsilon_x) - \frac{C^3}{- C - 2\epsilon_x}$$

(The product of the two imaginary roots is expressed in terms of the negative triple product of the roots - C^3 divided by the real roots.)

Using series expansion, and neglecting the terms which are higher than second order in $2\epsilon_x/C$

$$B \approx - C^2 - 4\epsilon_x C - 4\epsilon_x^2 + C^2(1 - \frac{2\epsilon_x}{C} + \frac{4\epsilon_x^2}{C^2}) \qquad \text{(A17)}$$

and

$$\epsilon_x \approx - \frac{B}{6C} \qquad \text{(A18)}$$

Similarly, the coefficient C^3 can be expressed as the product of the roots as follows

$$- C^3 = (- C - 2\epsilon_x) \left[(1/2C + \epsilon_x)^2 + (1/2 \cdot 3\ C - \epsilon_y)^2\right] \qquad \text{(A19)}$$

$$- C^3 \approx - C^3 + C^2(- 3\epsilon_x + \sqrt{3}\ \epsilon_y)$$

$$\epsilon_y \approx 3\ \epsilon_x \qquad \text{(A20)}$$

In terms of the stability derivatives we have

$$\epsilon_x \approx \frac{\epsilon_y}{\sqrt{3}} \approx \frac{1}{6} \frac{(1/3(M_\theta + X_u)^2 - M_\theta X_u}{(gM_u)^{1/3}}$$

Finally

$$\arctan \frac{\epsilon_y}{\epsilon_x} \approx 60^\circ \tag{A21}$$

These results indicate that the complex roots of the exact equation (A13) lie on lines inclined 60° with respect to the horizontal and pass through the complex roots of the approximating equation (A14). The distance between each of these roots and the related roots of the exact equation (A13) is approximately equal to $2\ |\epsilon_x|$.

These conclusions yield a new and quick method for estimating the roots of a cubic equation of the type given by (A13).

APPENDIX III

ILLUSTRATIONS

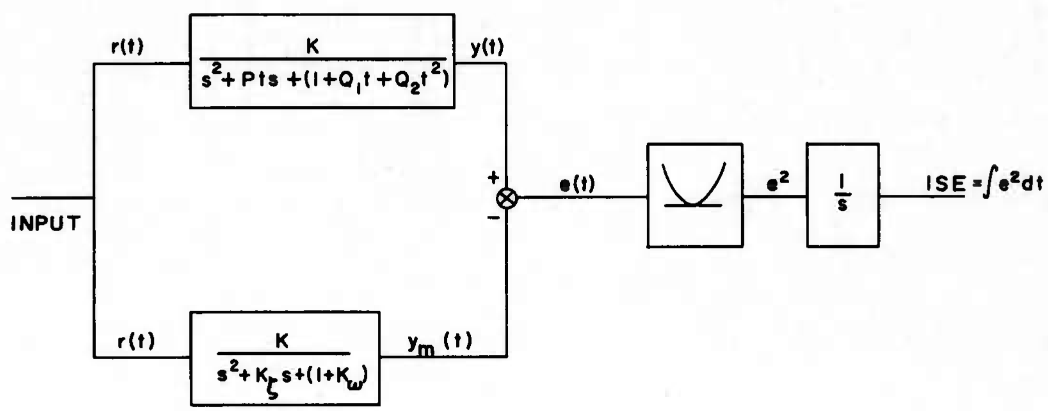


FIGURE 1. BLOCK DIAGRAM OF COMPUTER SETUP

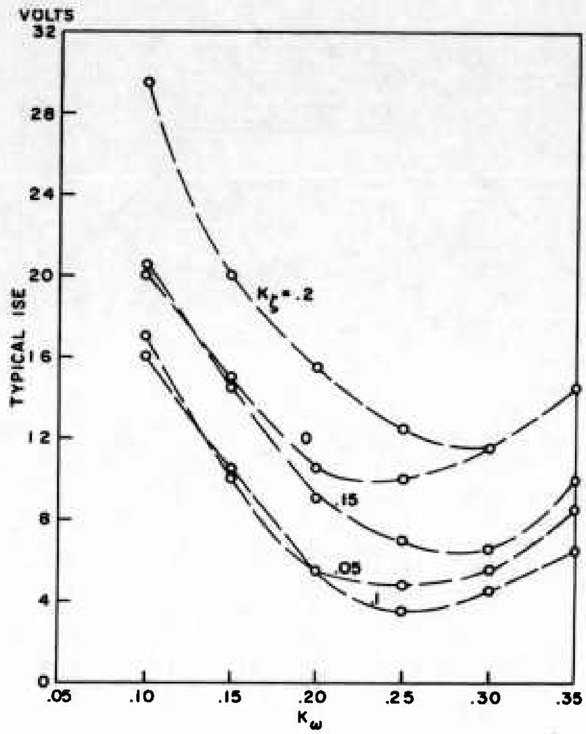


FIGURE 2. TYPICAL ISE SELECTIVITY

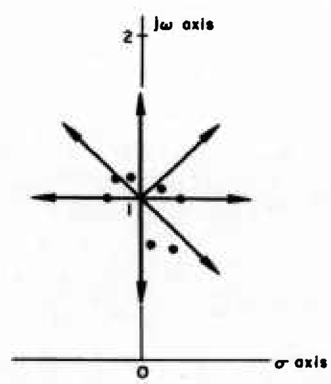


FIGURE 3. APPROXIMATING ROOTS

SPEED OF VARIATION: $\sqrt{Q^2 + P^2} = \frac{1}{4\pi}$

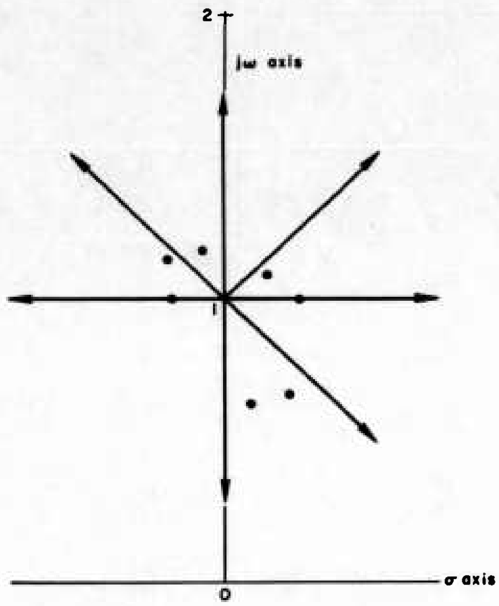


FIGURE 4. APPROXIMATING ROOTS

SPEED OF VARIATION: $\sqrt{Q^2+P^2} = \frac{1}{2\pi}$

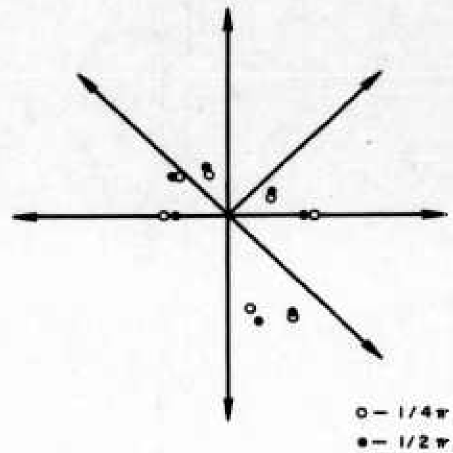


FIGURE 5. NORMALIZED APPROXIMATING ROOTS

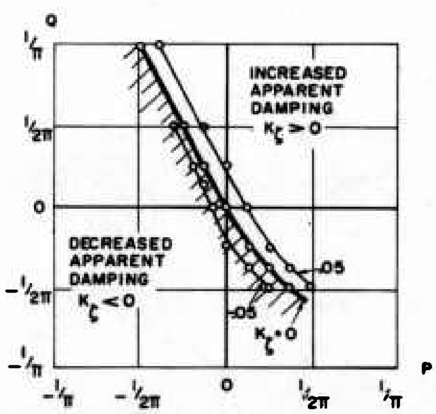


FIGURE 6. BOUNDARY BETWEEN INCREASED AND DECREASED APPARENT DAMPING

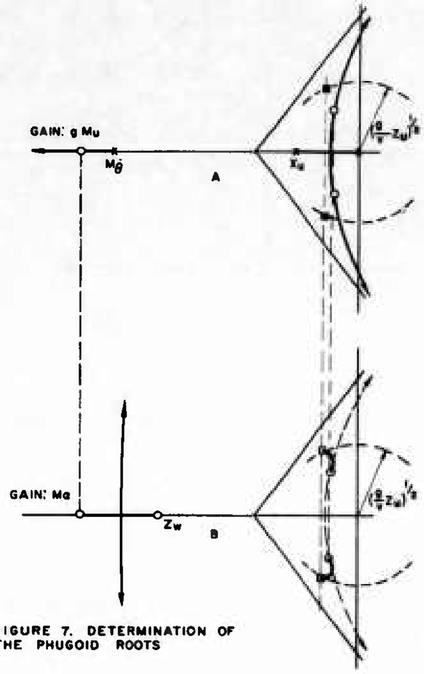


FIGURE 7. DETERMINATION OF THE PHUGOID ROOTS

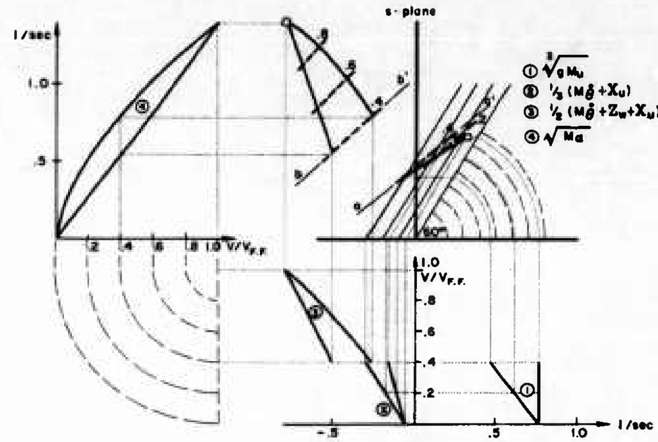


FIGURE 8. OSCILLATORY ROOTS DURING TRANSITION (NEAR-HOVERING AND NEAR-FORWARD-FLIGHT APPROXIMATION)

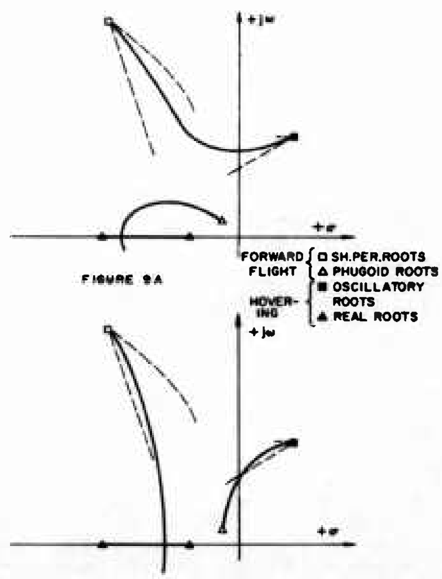


FIGURE 9A

FIGURE 9B

FIGURE 9. TWO POSSIBLE ROOT TRANSITIONS

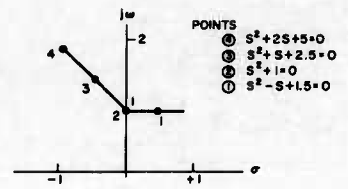


FIGURE 10A TRANSITION LOCUS

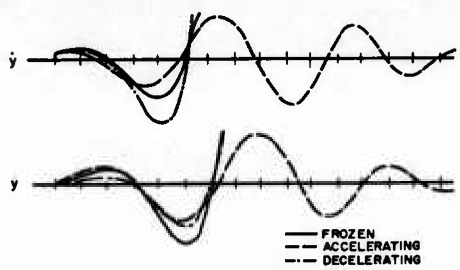


FIGURE 10B. TRANSIENTS IN POINT 1

FIGURE 10. 32 SECOND TRANSITION

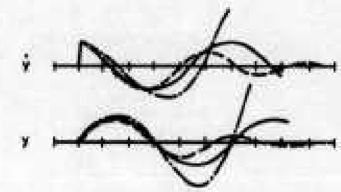


FIGURE 10C TRANSIENTS IN POINT 2

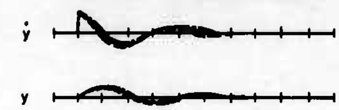


FIGURE 10D TRANSIENTS IN POINT 3

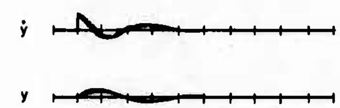


FIGURE 10E TRANSIENTS IN POINT 4

— FROZEN
— — ACCELERATING
—·— DECELERATING

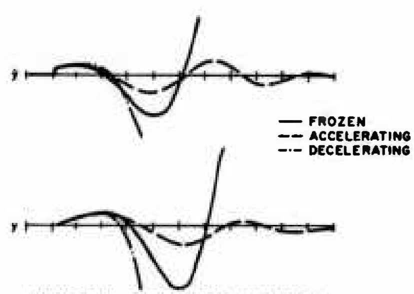


FIGURE 11A. TRANSIENTS IN POINT 1

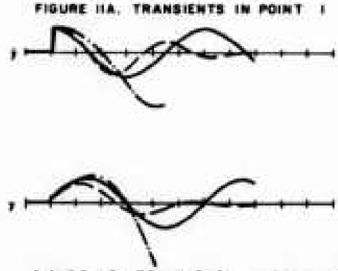


FIGURE 11B. TRANSIENTS IN POINT 2

FIGURE 11. 16 SECOND TRANSITION

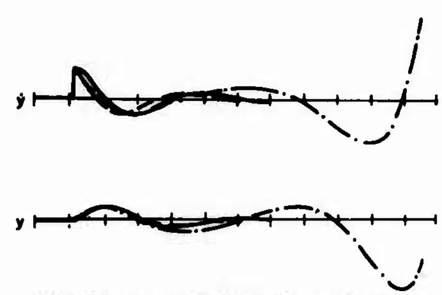


FIGURE 11C. TRANSIENTS IN POINT 3

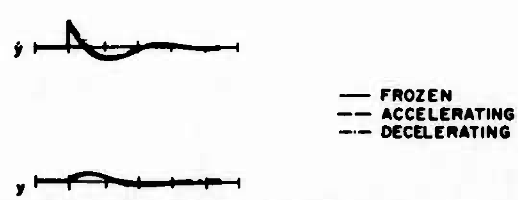


FIGURE 11D. TRANSIENTS IN POINT 4

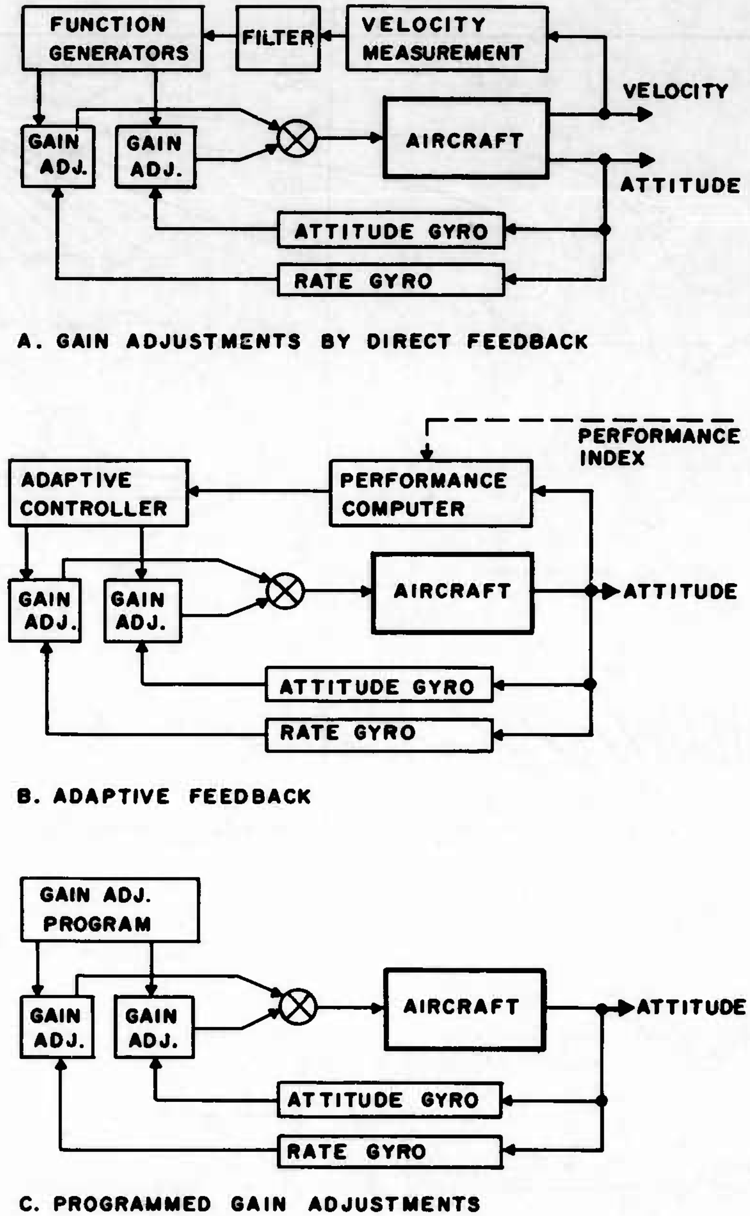
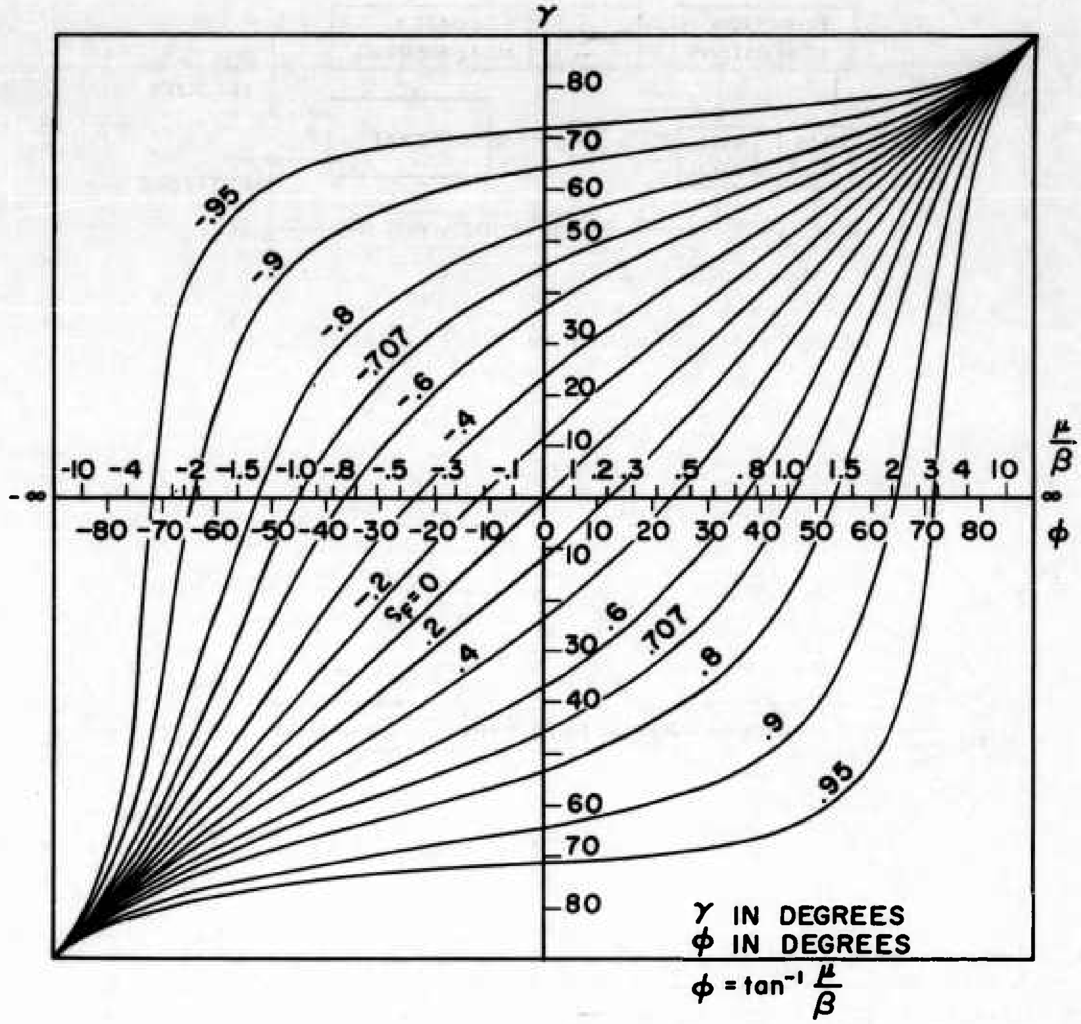
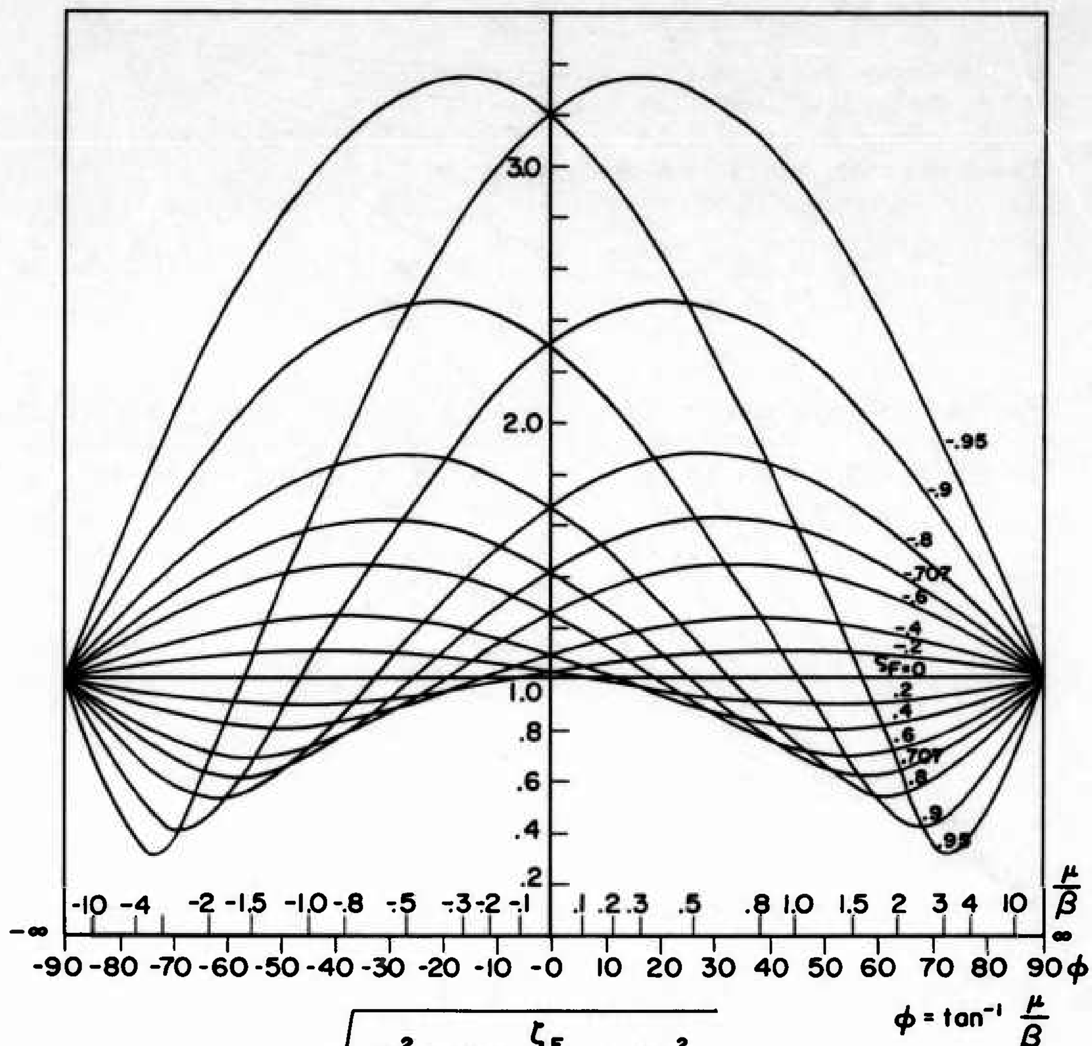


FIGURE 12. VARIABLE FEEDBACK CONFIGURATIONS



$$\gamma = \tan^{-1}\left(\frac{\mu}{\beta} - \frac{\zeta_F}{\sqrt{1-\zeta_F^2}}\right)$$

FIGURE 13. SLOPE OF FROZEN LOCUS



$$\frac{\sqrt{\beta^2 + \left(\mu - \frac{\zeta_F}{\sqrt{1-\zeta_F^2}}\,\beta\right)^2}}{\sqrt{\beta^2 + \mu^2}} \quad \text{VS. } \frac{\mu}{\beta}$$

FIGURE 14. NORMALIZED RATE OF TIME VARIATION

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mode of a system. The rate of time variation is described in terms of the deviation from the frozen system approximation. An analog computer study was made to specify quantitatively those rates of time variation which cannot be considered as slow.

The longitudinal dynamics of VTOL aircraft is studied as an example. Approximations and the application of root locus methods in terms of the most significant stability derivatives lead to a construction describing the behavior of the oscillatory roots during transition. The results are used in a discussion of the following variable feedback configurations: direct feedback adjustments, adaptive feedback, and programmed feedback adjustments.

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